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# Mirror mirror on the wall (On two-dimensional black holes and Liouville theory)

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Abstract: We present a novel derivation of the duality between the two-dimensional Euclidean black hole and supersymmetric Liouville theory. We realise these (1+1)-dimensional conformal field theories on the worldvolume of domain walls in a (2+1)-dimensional gauge theory. We show that there exist two complementary descriptions of the domain wall dynamics, resulting in the two mirror conformal field theories. In particular, effects which are usually attributed to worldsheet instantons are captured by the classical scattering of domain walls.

Keywords: Field Theories in Lower Dimensions, Solitons Monopoles and Instantons, 2D Gravity.



 

## Contents



## 1. Introduction

The two-dimensional black hole is a much studied object. It was originally introduced in the early '90s as a classical solution of two-dimensional string theory [1]. In its lorentzian form, the metric has the structure of the Schwarzchild solution and displays an event horizon. In the dark days before D-branes, this background was studied as a toy model to explore stringy properties of black holes. In these enlightened modern times, attention has focused on the Euclidean incarnation of the two-dimensional black hole which has the geometry of a semi-infinite cigar and is shown in figure 1. Of particular interest are the various cameo roles that this background plays in ten-dimensional string theory: it appears as a Calabi-Yau manifold develops an isolated singularity [2]; in the near-horizon limit of non-extremal NS5-branes [3]; and in the double-scaling limit of little string theory [4]. At the end of this paper we will extend this list by demonstrating how the black hole sigma model arises in the D2-D6 system of IIA string theory.

The black hole may be described as a $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset model at level $k$. The semi-infinite cigar of figure 1 has a metric and dilaton given by

$$\begin{aligned} ds^2_{BH} &= k[d\rho^2 + \tanh^2\rho\, d\theta^2] \\ \Phi &= \Phi_0 - 2\log\cosh\rho\,. \end{aligned} \tag{1.1}$$

The geometry is non-singular at the tip $\rho = 0$ if the coordinate $\theta$ is taken to have periodicity $2\pi$. The non-vanishing curvature close to the tip is compensated by the dilaton profile $\Phi$ to ensure one-loop conformal invariance. In the asymptotic regime $\rho \to \infty$ the string becomes weakly coupled and propagates in a cylindrical geometry of radius $\sqrt{k}$.

For large $k$ the radius of the asymptotic circle is large and one may employ semi-classical techniques to study the $(1+1)$-dimensional conformal field theory. For small $k$ one should attempt to find a T-dual description. It was conjectured by Fateev, Zamolodchikov and Zamolodchikov that this T-dual description is a Landau-Ginzburg model with a particular potential, known as the sine-Liouville theory [5]. This duality was used in [6] as the starting point in the construction of a matrix model for the black hole.



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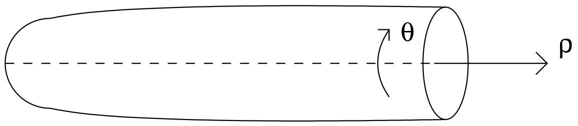


**Figure 1:** The geometry of the two-dimensional Euclidean black hole.

While much evidence for the duality exists [5, 6], a proof is currently lacking. Progress can be made through the introduction of supersymmetry. A Kazama-Suzuki supercoset construction results in a theory with the bosonic background described by equation (1.1) and $\mathcal{N} = (2,2)$ superconformal symmetry. The duality conjecture now states that the superconformal theory with target space (1.1) is dual to super-Liouville theory [4] (see also [7] for earlier discussions),

$$\mathcal{L} = \int d^4\theta \, \frac{1}{2k}|Y|^2 + \frac{\mu}{2}\left(\int d^2\theta \, e^{-Y} + \text{h.c.}\right). \tag{1.2}$$

Here $Y$ is a chiral superfield whose imaginary component has period $2\pi$, and $\mu$ is a mass scale. This action is also accompanied by a linear dilaton. The asymptotic regime of this theory $\text{Re}(Y) \to \infty$ describes a cylinder of radius $1/\sqrt{k}$, as befits the T-dual of the black hole. However, the opposite regime $\text{Re}(Y) < 0$ is disfavoured in the Liouville theory by the exponential rising potential. This is qualitatively different from the cigar metric, where the corresponding regime $\rho < 0$ is removed by the geometry.

A proof of the equivalence between the theories described by (1.1) and (1.2) was given by Hori and Kapustin[1] [8]. They realise the black hole background (1.1) as the infra-red fixed point of a gauged-linear sigma model. One advantage of this construction is that it introduces instantons into the picture in the guise of Nielsen-Olesen vortices. These appear despite the lack of two-cycles in the target space (a related discussion of this phenomenon in the presence of NS5-branes was given in [10]). Hori and Kapustin show that the effect of these instantons is to generate the superpotential (1.2) upon applying T-duality.[2]

In this paper we present another derivation of the duality, which has a very different flavour to previous proofs. Our hope is that these techniques may prove useful beyond the situation considered here.

The starting point is a $(2+1)$-dimensional abelian-Higgs model, with a mass gap and several isolated vacua. Our interest will be focused on the system of $(1+1)$-dimensional domain walls which interpolate between these vacua. The basic idea is that there are two different descriptions of the dynamics of the domain walls. In the first description, one studies domain walls in the classical theory and subsequently quantises their zero modes. In the second description, one integrates out the heavy modes in three dimensions, and then examines the dynamics of classical domain walls in the effective theory. We shall see that these two descriptions are precisely the two mirror theories (1.1) and (1.2).

[1]Their proof follows closely the techniques of [9] prompting them to refer to this duality as "mirror symmetry". This, in turn, made the title of the current paper sadly unavoidable.

[2]A second derivation of this duality was given in [11] by compactifying Chern-Simons mirror theories [12] from three dimensions. In this case the instanton effects in two-dimensions are related to one-loop effects in three-dimensions.

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The results of this paper are somewhat reminiscent of $\mathcal{N}=4$ super-Yang-Mills theories in $d=(2+1)$ dimensions [13]. Recall that the instanton effects of three-dimensional gauge theories (which are monopole configurations) are captured by the classical dynamics of monopoles in a different gauge group.[3] In the present setting, we have instanton effects of a two-dimensional gauge theory (which are Nielsen-Olesen vortices) once again captured by the classical dynamics of solitons: in this case domain walls.

The paper is organised as follows. In the following section 2 we present a pair of domain walls which feel no static force. We show that the velocity dependent interactions between the walls are described by a non-linear sigma-model with target space (1.1). In section 3 we study the same domain walls in an effective, low-energy three-dimensional theory. We find that the quantum effects induce a static force between the two domain walls which is described by the Liouville theory (1.2). In section 4, we discuss several further aspects of this idea, including a brane construction in the D2-D6 system of IIA string theory and the realisation of other toric sigma models — such as the $\mathbf{CP}^n$ model — on the worldvolume of domain walls.

## 2. The black hole from domain wall dynamics

Our starting point is a $d=(2+1)$ dimensional U(1) gauge theory with $\mathcal{N}=4$ supersymmetry (8 supercharges). For interesting domain wall dynamics, we need three or more isolated vacua, resulting in two or more domain walls. We choose the simplest case of three vacua, which requires three charged hypermultiplets. The bosonic part of the lagrangian is given by

$$\begin{aligned}\mathcal{L} = \frac{1}{4e^2}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2e^2}|\partial\boldsymbol{\phi}|^2 + \sum_{i=1}^{3}(|\mathcal{D}q_i|^2 + |\mathcal{D}\tilde{q}_i|^2) - \sum_{i=1}^{3}|\boldsymbol{\phi}-\mathbf{m}_i|^2(|q_i|^2+|\tilde{q}_i|^2) - \\ - \frac{e^2}{2}\left|\sum_{i=1}^{3}\tilde{q}_iq_i\right|^2 - \frac{e^2}{2}\left(\sum_{i=1}^{3}|q_i|^2 - |\tilde{q}_i|^2 - \zeta\right)^2 .\end{aligned} \tag{2.1}$$

Here $\boldsymbol{\phi}$ is a triplet of neutral scalar fields which live in the vector multiplet. Each hypermultiplet contains two complex scalar fields, $q_i$ and $\tilde{q}_i^\dagger$, with charge $+1$ under the U(1) gauge group. In three dimensions, each hypermultiplet is assigned a triplet of masses $\mathbf{m}_i$.

If the masses $\mathbf{m}_i$ are distinct, and the FI parameter is strictly positive $\zeta>0$, then the theory has three isolated, massive vacua, given by

$$\text{Vacuum } i: \qquad \boldsymbol{\phi}=\mathbf{m}_i\,, \qquad |q_j|^2=\zeta\,\delta_{ij}\,, \qquad |\tilde{q}_j|^2=0 \qquad i=1,2,3\,. \tag{2.2}$$

Let us examine the vacuum physics. In each ground state, supersymmetry is unbroken and the theory is in a Higgs phase. Together with the broken U(1) gauge symmetry, the theory also exhibits an unbroken $U(1)_F^2$ flavour symmetry, acting on the $q_i$ and $\tilde{q}_i^\dagger$. All three vacua have a mass gap, with the lightest excitations depending on the ratio $k\sim\zeta/\Delta\mathbf{m}_i$ where $\Delta\mathbf{m}_i$ are the mass splittings. In the limit $k\to\infty$, hypermultiplet modes become massless,

[3] For a quantitative comparison of instanton computations vs. monopole scattering, see [14]

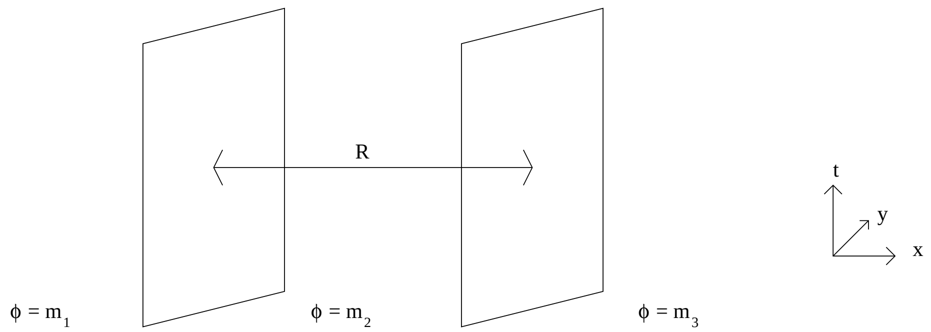


**Figure 2:** Two domain walls. (No, really). The separation $R$ is a modulus.

reflected by the appearance of a Higgs branch. In contrast, as $k \to 0$, the vector multiplet becomes light and a Coulomb branch of vacua appears. The parameter $k$ will be defined more precisely below, and plays an important role in our story.

**Classical domain wall dynamics.** The system of $\frac{1}{2}$-BPS domain walls that interpolate between these isolated vacua has been studied in [15]–[18]. The tension of a domain wall interpolating from the $i^{\rm th}$ to the $j^{\rm th}$ vacuum is given by $T_{ij} = \zeta|\mathbf{m}_j - \mathbf{m}_i|$. This suggests that something special may happen when we choose the mass triplets to be co-linear $\mathbf{m}_i = (m_i, 0, 0)$ with, say, $m_{i+1} > m_i$. For in this case, we have

$$T_{13} = T_{12} + T_{23} \tag{2.3}$$

which is a necessary, although not sufficient, condition for there to be no classical force between a domain wall interpolating from the $1^{\rm st}$ to the $2^{\rm nd}$ vacuum, and a domain wall interpolating from the $2^{\rm nd}$ to the $3^{\rm rd}$ vacuum. In fact this energetic reasoning is correct and there do exist static solutions corresponding to domain walls with arbitrary separation $R$ as shown in figure 2.

To see that this is the case, we examine the zero modes structure of solutions to the Bogomoln'yi equations describing these domain walls. These equations, first derived in [15], involve only $\boldsymbol{\phi} = (\phi, 0, 0)$,

$$\begin{aligned} \partial\phi &= e^2\left(\sum_{i=1}^{3}|q_i|^2 - \zeta\right) \\ \mathcal{D}q_i &= (\phi - m_i)q_i\,, \end{aligned} \tag{2.4}$$

where $\partial\phi$ denotes the derivative of $\phi$ with respect to the spatial coordinate $x$ transverse to the domain wall, and $\mathcal{D}q = \partial q - iA_x q$ is the covariant derivative. In the strong coupling limit $e^2 \to \infty$, it was shown in [16, 17] that these equations do admit a moduli space of solutions, with one of the collective coordinates having the interpretation of the separation between the walls. At finite $e^2$, an index theory computation was performed by Lee [18], revealing that solutions to (2.4) indeed have the relevant zero modes.

Let us describe these zero modes in more detail. Each of the walls has two collective coordinates. One of these is simply the position of the domain wall, while the second is an internal, periodic degree of freedom which comes from acting on the domain wall solution with one of the $U(1)_F$ flavor symmetries [19]. These are analogous to the collective coordinates of monopoles that arises from large gauge transformations. Our system of two domain walls therefore has 4 collective coordinates, corresponding to their center of mass, their separation, and 2 internal phases. Furthermore, there are 8 fermionic zero modes, 4 of which arise from broken supersymmetry while the remaining ones are guaranteed by the unbroken $\mathcal{N} = (2, 2)$ supersymmetry preserved on the worldvolume of the domain walls.

The low-energy dynamics of the domain walls is thus described as a $d = (1 + 1)$-dimensional sigma model with $\mathcal{N} = (2, 2)$ supersymmetry, and with target space given by the domain wall moduli space $\mathcal{M}$. The simplest case to consider is when the two domain walls have equal tension. This occurs when the mass parameters are given by $m_i = (-M/2, 0, M/2)$ which means that $T_{12} = T_{23} = \zeta M/2$. In this case, the moduli space of two domain walls has the structure [16],

$$\mathcal{M} = \mathbf{R} \times \frac{\mathbf{S}_1 \times \tilde{\mathcal{M}}}{\mathbf{Z}_2}, \tag{2.5}$$

where the $\mathbf{R}$ factor is parameterised by $r$, the center of mass of the domain walls, while the $\mathbf{S}^1$ factor is parameterised by $\tau \in [0, 2\pi)$, the overall phase of the domain walls. All interesting information about the dynamics of the domain walls is contained in the relative moduli space $\tilde{\mathcal{M}}$. This two-dimensional manifold is parameterised by a variable $R \in (-\infty, \infty)$ and the relative phase $\theta \in [0, 2\pi)$. For large values, $R$ is equal to the separation between the domain walls. However at distances less than $(\zeta M)^{-1/2}$, when the domain wall cores overlap, this interpretation breaks down as is obvious from the fact that $R$ takes negative values. The $\mathbf{Z}_2$ action acts as $\chi \to \chi + \pi$ and $\theta \to \theta + \pi$. The metric on $\mathcal{M}$ was calculated in [17] and is given by,

$$ds^2_{DW} = k\left[dr^2 + d\tau^2 + e^{\psi_{DW}(R)}\left(\frac{M^2}{16}dR^2 + d\theta^2\right)\right]. \tag{2.6}$$

The overall scale of the metric is

$$k = \frac{2\zeta}{M} \tag{2.7}$$

and will soon be identified with the level $k$ of the coset space construction of the 2D black hole (1.1). To compare to the black hole metric, it will prove convenient to measure the separation between domain walls in terms of the dimensionless quantity,

$$u = \frac{RM}{4}.$$

All information about the classical scattering of domain walls is encoded within the smooth, real, and somewhat ugly function $\psi_{DW}(u)$, given by [17]

$$e^{\psi_{DW}(u)} = \frac{e^{2u}}{e^{4u} - 4}\left(e^{2u} - \frac{4}{\sqrt{4 - e^{4u}}}\cos^{-1}\left(\frac{e^{2u}}{2}\right)\right).$$

Although somewhat hidden in these coordinates, the relative moduli space of the domain walls does look like the semi-infinite cigar of figure 1. This is obvious in the limit $u \to \infty$,



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where $e^{\psi(u)} \to 1$, so that the moduli space becomes cylindrical. It is less obvious that the moduli space is cigar-like at $u \to -\infty$, but this is made more plausible by noting that the point at $-\infty$ is at finite affine distance in the metric (2.6). A suitable coordinate transformation brings this point to finite parametric distance, where it can be checked that the tip of the cigar is smooth [17]. Rather than take this route, we will instead bring the black hole metric into the form of (2.6). Define $\sinh\rho = e^u$, so that the metric (1.1) becomes,

$$ds^2_{BH} = ke^{\psi_{BH}(u)}\left(du^2 + d\theta^2\right), \tag{2.8}$$

where

$$e^{\psi_{BH}(u)} = \frac{e^{2u}}{1+e^{2u}}\,.$$

Note that both $\psi_{DW}(u)$ and $\psi_{BH}(u)$ have the same asymptotic behaviour,

$$\psi(u) \to \begin{cases} 0 & u \to +\infty \\ 2u & u \to -\infty \end{cases}\,. \tag{2.9}$$

Away from this asymptotic regime, the metrics differ.

**Quantum domain wall dynamics.** Neither the domain wall metric (2.6), nor the black hole metric (2.8), are Ricci flat. This non-zero curvature gives a one-loop contribution to the beta-function of the form $\beta_{ij} = -R_{ij}/2\pi$. In the absence of any other sources, the metric therefore changes with scale $t$. For target spaces of the cigar form the RG equations read,

$$ke^{\psi}\frac{\partial\psi(u,t)}{\partial t} = \frac{1}{4\pi}\frac{\partial^2\psi(u,t)}{\partial u^2}\,. \tag{2.10}$$

In the two-dimensional black hole solution (1.1), the term on the right-hand side is canceled by the contribution from the dilaton. As shown by Hori and Kapustin [8], one can mimic the dilaton contribution by assigning an anomalous transformation to the target space coordinates under Weyl rescaling. The basic idea is that the curvature at the tip of the cigar causes the cigar to shrink. To keep things looking conformal, we should run along with this shrinking tip. Under the condition that the function $\psi(u)$ obeys the boundary conditions (2.9), then it can be shown [8] that the existence a conformal fixed point requires the anomalous Weyl transformation,

$$u \to u + \frac{t}{2\pi k}\,.$$

From the perspective of the domain walls, as an observer examines the physics on larger and larger distance scales ($t \to \infty$) she must increase the distance between the domain walls in an attempt to keep things looking the same. Writing $\psi$ as a function of the scale invariant quantity $v = u - t/2\pi$, the RG equations become

$$ke^{\psi}\frac{\partial\psi(v,t)}{\partial t} = \frac{1}{4\pi}\frac{\partial^2\psi(v,t)}{\partial v^2} + \frac{e^{\psi}}{2\pi}\frac{\partial\psi(v,t)}{\partial v}\,.$$

It is simple to check that the unique conformal solution ($\dot{\psi} = 0$) to this equation is the black hole metric (2.8), where second term on the right-hand-side has played the role of the



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dilaton. We are now in a position to see how the domain wall metric (2.6) evolves under the RG group. It is a simple matter to check numerically that the domain wall metric does indeed flow towards the conformal black hole metric (2.8) in the infra-red.

The one-loop approximation to the beta-function requires large $k$, and our derivation of the black hole conformal field theory is therefore valid only in this limit. It is possible to go beyond this approximation using a further result of Hori and Kapustin [8]: there is a unique SCFT with the asymptotic behaviour and symmetries of the black hole and central charge,

$$c = 3\left(1 + \frac{2}{k}\right)$$

which is indeed the central charge of the Kazama-Suzuki $\mathrm{SL}(2,\mathbb{R})/\,\mathrm{U}(1)$ coset construction at level $k$. Thus to establish the connection to the black hole SCFT for all values of $k$, we must determine the infra-red central charge of the domain wall dynamics. This remains an open problem.

**Losing supersymmetry.** The above discussion has, for the most part, not relied on the supersymmetry of the three-dimensional field theory. Of course, the three dimensional fermions supply the zero modes which ensure that we end up with $\mathcal{N} = (2,2)$ supersymmetry on the domain wall worldvolume, but these fermions affect neither the classical metric (2.6) nor the RG flow equations (2.10). We could happily start from the three-dimensional bosonic lagrangian (2.1), omitting the fermions, and arrive at the same low-energy description of the domain wall dynamics in terms of the bosonic CFT on the black hole background (1.1). The only deviation from the formulas above is that the central charge is now expected to be [1]

$$c_{\text{bosonic}} = \frac{3k}{k-2} - 1\,.$$

Curiously, for the purely bosonic black hole, the coset construction only makes sense for $k > 2$. From our perspective, this requires that classically $\zeta > M$. It would be interesting to understand this restriction in terms of phase transitions of the three-dimensional field theory. When quantum effects are taken into account, it seems plausible that the bosonic theory exits the gapped Higgs phase and enters a massless phase around $\zeta \sim M$.

## 3. Liouville theory from domain wall dynamics

In this section we present a different description of the domain wall dynamics, valid when $k = 2\zeta/M \ll 1$. To see what this new description may involve, note that in the limit $k \to 0$ the vector multiplet fields become light. This suggests that we should integrate out the hypermultiplets, leaving behind an effective lagrangian for the photon, the triplet of neutral scalar fields $\boldsymbol{\phi}$, and their fermionic partners. This technique of examining theories for different values of $k$ was employed in the context of supersymmetric quantum mechanics in [24].

One important point is that in three dimensions we can dualise the photon in favour of a periodic scalar field $\sigma$ defined as $F = {}^\star d\sigma$. The low-energy effective action is given by a massive hyperkähler sigma-model preserving $\mathcal{N}=4$ supersymmetry, with the bosonic interactions described by [25]

$$\mathcal{L}=\frac{1}{2}H(\boldsymbol{\phi})\,|\partial\boldsymbol{\phi}|^2+2H(\boldsymbol{\phi})^{-1}(\partial\sigma+\boldsymbol{\omega}\cdot\partial\boldsymbol{\phi})^2-\frac{1}{2}\zeta^2H(\boldsymbol{\phi})^{-1}\,, \tag{3.1}$$

where $\boldsymbol{\omega}$ is defined as $\nabla\times\boldsymbol{\omega}=\nabla H$, and

$$H(\boldsymbol{\phi})=\frac{1}{e^2}+\sum_{i=1}^{3}\frac{1}{|\boldsymbol{\phi}-\mathbf{m}_i|}\,. \tag{3.2}$$

The first two terms in (3.1) describe a sigma-model with the target space given by the Gibbons-Hawking metric with three centers. Notice in particular the interaction term between the dual photon $\sigma$ and the scalars $\boldsymbol{\phi}$ which arises from integrating out fermions in the hypermultiplet: we shall return to the importance of this coupling shortly. The final term in (3.1) is a potential, lifting the Coulomb branch. It is the unique potential allowed by $\mathcal{N}=4$ supersymmetry [26]. The lagrangian (3.1) preserves the vacuum structure of the classical theory (2.2), with three supersymmetric ground states given by

$$\text{Vacuum } i: \qquad \boldsymbol{\phi}=\mathbf{m}_i \qquad i=1,2,3\,.$$

In each of these vacua, the kinetic terms in (3.1) are singular, but this is simply a coordinate singularity and may be eliminated by a field redefinition in which $\sqrt{\phi}$ is treated as the canonically normalised field.

**Classical domain wall dynamics.** Solitons in massive sigma-models of this type have been much studied in the literature [16, 17] and [19]–[23]. As for the classical theory of the previous section, the BPS domain walls have tension $T_{ij}=\zeta|\mathbf{m}_i-\mathbf{m}_j|$. Restricting once again to co-linear masses, $\mathbf{m}_i=(m_i,0,0)$, the BPS equations describing domain walls are given by

$$\partial\phi-\zeta\,H(\phi)^{-1}=0 \tag{3.3}$$
$$\partial\sigma+\boldsymbol{\omega}\cdot\partial\boldsymbol{\phi}=0\,. \tag{3.4}$$

Let us start by examining the first of these equations. We are interested in solutions which interpolate between the first vacuum at $\boldsymbol{\phi}=\mathbf{m}_1$ and the third vacuum at $\boldsymbol{\phi}=\mathbf{m}_3$. However, equation (3.3) involves only a single scalar field $\boldsymbol{\phi}=(\phi,0,0)$ and it is well known that such systems admit solutions only for domain walls interpolating between neighbouring vacua. There is no solution interpolating between the first and third vacua. One could of course attempt to construct an approximate solution by considering a configuration of two, well-separated domain walls. But, as we shall review in detail below, such a configuration results in a repulsive force between the two domain walls. This force occurs despite the fact that each wall preserves (the same) half of the eight supersymmetries in three dimensions and therefore provides a counterexample to the commonly stated maxim that BPS necessarily implies no force (for a discussion of this system, see [16]).

The above results are in stark contrast to the previous section where we saw that there exist solutions in the classical theory corresponding to domain walls with arbitrary separation. In that case, the existence of these solutions resulted in a description of the quantum dynamics of the domain walls in terms of a sigma-model. In the present case, the existence of a potential means that the description of the domain wall dynamics will be in the form of a Landau-Ginzburg theory. In the remainder of this section, we shall calculate the force experienced by two well-separated domain walls, and show that their dynamics is governed by the Liouville theory (1.2).

**The force between domain walls.** A technique for computing the low-energy dynamics describing the repulsive force between domain walls was presented long ago by Manton [27], and discussed more recently in [28]. Manton's method is a beautifully direct and simple approach to the problem, and proceeds as follows: One firstly constructs an approximate solution to the equations of motion consisting of a superposition of two well-separated domain walls, suitably patched together in the middle. One then examines how this configuration evolves in time under the full second order equations of motion and calculates the acceleration of the part of the field configuration describing just one of two kinks. The magic of the approach lies in the fact that this acceleration is independent of exactly where one chooses to do the patching. Let us now see in detail how to apply Manton's procedure to our situation.

To start, we wish to construct an approximate solution consisting of two well-separated domain walls. We will be aided in this task by knowing the solution for a single domain wall so let us begin with this. For clarity, we will keep the masses $m_i$ arbitrary, subjected only to $\sum_i m_i = 0$. Only later will we restrict to domain walls of equal tension. Consider first the kink which interpolates between the first and second vacua, with tension $T_{12} = \zeta(m_2 - m_1)$. Throughout space the field $\phi$ lies within the range $m_1 \leq \phi \leq m_2 < m_3$, and the domain wall equations therefore read,

$$\text{kink } 1 \to 2: \qquad \partial\phi = \zeta\left(\frac{1}{e^2} + \frac{1}{\phi - m_1} + \frac{1}{m_2 - \phi} + \frac{1}{m_3 - \phi}\right)^{-1}.$$

For finite $e^2$, one can reduce this to an algebraic equation for $\phi$. However, in the limit $e^2 \to \infty$, an explicit solution is known [19]. If we take the domain wall to be centered at $x = x_1$, then the kink profile $\phi_{12}(x - x_1)$ is given by the solution to the quadratic equation,

$$\text{kink } 1 \to 2: \qquad e^{\zeta(x-x_1)}(m_2 - \phi_{12})(m_3 - \phi_{12}) - m_3(\phi_{12} - m_1) = 0\,,$$

where, in order to get the correct boundary conditions, we must take the negative square root. It will prove useful to have the asymptotic form of the profile to the far right of the kink. As $(x - x_1) \to \infty$, we have

$$\text{kink } 1 \to 2: \qquad \phi_{12} \;\to\; m_2 - m_2\frac{m_2 - m_1}{m_3 - m_2}e^{-\zeta(x-x_1)} + \mathcal{O}\left(e^{-2\zeta(x-x_1)}\right). \qquad (3.5)$$

There is a similar story for the kink interpolating between the second and third vacua with tension $T_{13} = \zeta(m_3 - m_2)$. This time $m_1 < m_2 < \phi < m_3$, so that the the equation of motion reads,

$$\text{kink } 2 \to 3: \qquad \partial\phi = \zeta\left(\frac{1}{e^2} + \frac{1}{\phi - m_1} + \frac{1}{\phi - m_2} + \frac{1}{m_3 - \phi}\right)^{-1}.$$



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The profile of the kink positioned at $x = x_2$ is given by the solution to the quadratic equation,

$$\text{kink } 2 \to 3: \qquad e^{-\zeta(x-x_2)}(\phi_{23} - m_1)(\phi_{23} - m_2) + m_2(m_3 - \phi_{23}) = 0\,.$$

This time we will be interested in the asymptotic regime far to the left of this kink. As $(x - x_2) \to -\infty$, the profile looks like

$$\text{kink } 2 \to 3: \qquad \phi_{23} \;\to\; m_2 - m_1\frac{m_3 - m_2}{m_2 - m_1}e^{\zeta(x-x_2)} + \mathcal{O}(e^{2\zeta(x-x_2)})\,. \tag{3.6}$$

We now wish to construct a configuration which looks like two far-separated domain walls. An obvious guess is to fix $x_2 - x_1$ to be large, and write $\phi_{13}(x) = \phi_{12}(x - x_1) + \phi_{23}(x - x_2) - m_2$, which has the correct boundary conditions. However, one can check that this configuration is actually singular where they join because of the coordinate singularity in the target space of (3.1) at the point $\phi = m_2$. The correct procedure to patch the two kinks together is to transform to a basis of fields which have canonical kinetic term at the point where the two kinks join $\phi = m_2$. This is achieved by setting

$$f(\phi_{13}(x) - m_2) = f(\phi_{12}(x - x_1) - m_2) + f(\phi_{23}(x - x_2) - m_2)\,, \tag{3.7}$$

where the function $f(\phi - m_2)$ is defined as

$$f(\phi - m_2) = \begin{cases} \sqrt{\phi - m_2} & \phi \geq m_2 \\ -\sqrt{m_2 - \phi} & \phi < m_2 \end{cases}\,.$$

The configuration $\phi_{13}(x)$ described by (3.7) is our initial condition for a profile describing two well-separated domain walls. The next step is to understand how this configuration evolves under the equations of motion of the system. After consistently truncating to the field of interest $\phi$, the equations of motion derived from (3.1) read,

$$\frac{1}{2}\frac{\partial H}{\partial \phi}\phi'^{\,2} + H\phi'' - \frac{1}{2}\zeta^2 H^{-2}\frac{\partial H}{\partial \phi} = 0 \tag{3.8}$$

We will consider the change in momentum experienced by the first kink. The momentum $P$ of the field between the point $x = a$ and the point $x = b$ is given by $T_{0x}$ component of the energy-momentum tensor,

$$P = -\int_a^b dx\; H(\phi)\dot{\phi}\phi'$$

from which we can compute rate of change of the momentum,

$$\begin{aligned} \dot{P} &= -\int_a^b dx\; \frac{\partial H}{\partial \phi}\dot{\phi}^2\phi' + H\ddot{\phi}\phi' + H\dot{\phi}\dot{\phi}' \\ &= \left[-\frac{1}{2}H(\dot{\phi}^2 + \phi'^{\,2}) + \frac{1}{2}\zeta^2 H^{-1}\right]_a^b\,, \end{aligned} \tag{3.9}$$

where we have made use of the equation of motion (3.8) in the second line to express $\dot{P}$ as a total derivative. We now evaluate (3.9) on our two-kink configuration (3.7), with the initial conditions $\dot{\phi}_{13} = 0$. Since we wish to compute the acceleration of the part of the field corresponding to the first kink, we set $a \to -\infty$ while choosing $b$ to lie somewhere between the two domain walls. We require only that both $(b - x_1)$ and $(x_2 - b)$ are large. We then expand the domain wall configuration (3.7) to leading order, using the expressions for the two kink solutions (3.5) and (3.6). Upon substituting these expressions into (3.9), the calculation offers up two minor miracles: the final answer is independent of $b$, and independent of the ubiquitous ratio $(m_2 - m_1)/(m_3 - m_2)$. We find,

$$\dot{P} = -2\zeta^2\sqrt{-m_1 m_3}\, e^{-\zeta(x_2 - x_1)/2}\,. \tag{3.10}$$

**The low-energy effective lagrangian.** Using Manton's method, we have computed the force experienced by the separated domain walls (3.10). Since each action has an equal and opposite reaction [New3], $\dot{P} = T_{12}\ddot{x}_1 = -T_{12}\ddot{x}_2$, from which we find the rate of change of the separation $R$ between the kinks,

$$\ddot{R} = \ddot{x}_2 - \ddot{x}_1 = 2\zeta\sqrt{-m_1 m_3}\frac{(m_3 - m_1)}{(m_2 - m_1)(m_3 - m_2)} e^{\zeta(x_2 - x_1)/2}(3.8) \tag{3.11}$$

Newton's first law is also useful, giving us the formula for the center of mass acceleration $\ddot{r} = \frac{1}{2}(\ddot{x}_2 + \ddot{x}_1) = 0$. At this point, we specialise to two domain walls of equal tension by setting the bare masses equal to $m_i = (-M/2, 0, M/2)$ as in section 2. The equation of motion (3.8) for the domain wall then follows from the $(1+1)$-dimensional effective action

$$\mathcal{L}_{\text{position}} = k\left[(\partial r)^2 + \frac{M^2}{16}(\partial R)^2\right] - 2\zeta M e^{-\zeta R/2}\,,$$

where we have covariantised the kinetic term to ensure the lagrangian is Lorentz invariant along the domain wall. The overall normalisation of this lagrangian is in agreement with (2.6) and is correct for the motion of a two domain walls, each of tension $T = M\zeta/2$.

So far we have considered collective coordinates associated to the position of the domain wall. As in the previous section, there exist further periodic, internal degrees of freedom for each wall. These arise from the second Bogomoln'yi equation (3.4). To account for these, we must firstly rewrite the triplet of scalars $\boldsymbol{\phi}$ in polar coordinates,

$$\boldsymbol{\phi} = (\phi, \xi\cos\chi, \xi\sin\chi)$$

in terms of which the second Bogomoln'yi equation (3.4) becomes

$$\sigma' + \sum_{i=1}^{3}\frac{\phi - m_i}{\sqrt{\xi^2 + (\phi - m_i)^2}}\chi' = 0\,, \tag{3.12}$$

where both $\sigma$ and $\chi$ have period $2\pi$. Thus, in the background of the first kink which interpolates from the first to second vacua, we have

$$\text{kink } 1 \to 2: \qquad \sigma' - \chi' = 0$$

while, in the background of the second kink, interpolating from the second to the third vacua,

$$\text{kink } 2 \to 3: \qquad \sigma' + \chi' = 0\,.$$

The relative minus signs between these two equations ensures the following, crucial fact: in the background of the two kink configuration (3.7) both the value of $\sigma$ and the value of $\chi$ remain good collective coordinates, and neither field appears in the potential. Their kinetic terms in the background (3.7) may be determined from the lagrangian (3.1) and are given by,

$$\mathcal{L}_{\text{phase}} = \frac{4}{k}\left((\partial\sigma)^2 + (\partial\chi)^2\right).$$

Comparing to the effective domain wall action that we found in section 2 (2.6), we see that $\sigma$ and $\chi$ are related to $\tau$ and $\theta$ respectively by T-duality. The final bosonic lagrangian is given by $\mathcal{L}_{\text{position}} + \mathcal{L}_{\text{phase}}$, from which we extract the part concerned with the relative motion of the domain walls involving the fields $R$ and $\chi$. An important observation is that this low-energy theory has a natural complex structure,

$$Y = \zeta\frac{R}{4} + i\chi$$

in terms of which the relative domain wall dynamics are described by,

$$\mathcal{L}_{\text{relative}} = 8\left(\frac{1}{2k}|\partial Y|^2 - \frac{\zeta M}{4}|e^{-Y}|^2\right). \tag{3.13}$$

Here the overall coefficient 8 has been factored out to ensure that the normalization of the periodic coordinate agrees with the bosonic terms from the Liouville theory (1.2). Having matched these kinetic terms, any non-trivial comparison with the Liouville model lies with the potential. The overall scale of the potential simply determines the subtraction point $\mu$ of the two-dimensional theory in terms of three dimensional parameters: $\mu = \zeta M/4$. This leaves us with only the coefficient of the exponent to check. Comparing the potential of (3.13) with the superpotential of (1.2), we see that the repulsive force between the two domain walls is indeed described by the Liouville theory (1.2).

**Losing supersymmetry.** Once again, we could examine how the above argument fares in the absence of supersymmetry. The conjecture is that if we start from the three-dimensional purely bosonic lagrangian (2.1), then the relative dynamics of domain walls are described by the sine-Liouville theory with lagrangian [5, 6]

$$\tilde{\mathcal{L}} = \frac{1}{k-2}(\partial y)^2 + \frac{1}{k}(\partial\chi)^2 - \mu^2 e^{-y}\cos\chi, \tag{3.14}$$

where $y$ is related to the separation of the walls. There are two key differences between this lagrangian and the super-Liouville theory. The first is that shifts of $\chi$ are no longer a symmetry in the bosonic case. The second is that the model is unstable for $k < 2$, mimicking the fact that the coset construction of the black hole only makes sense for $k > 2$.

Can we reproduce (3.14) from domain wall dynamics? Unfortunately, and in contrast to section 2, supersymmetry played a vital role in determining the domain wall dynamics in this section. Without the strong holomorphic (in fact, hyperKähler) restrictions imposed by supersymmetry on the low-energy three-dimensional effective action (3.1), we have no

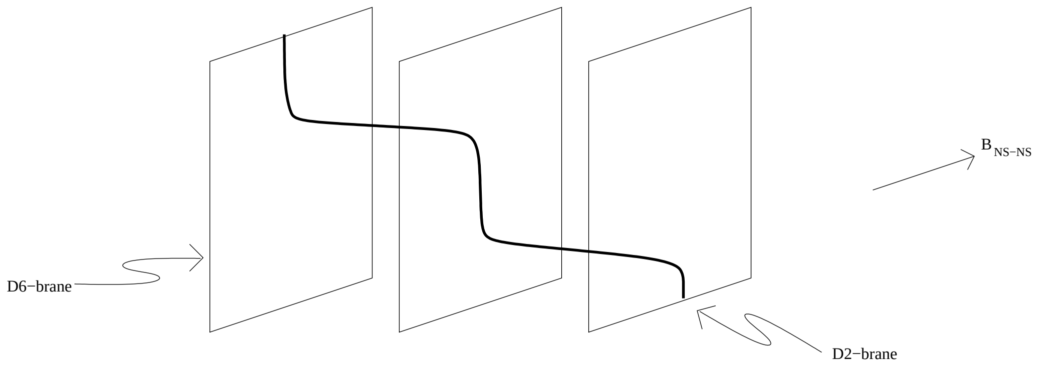


**Figure 3:** Two domain walls in the D2-D6 system. In contrast to other pictures, the sheets represent the D6-branes. The vacua of the field theory occur when the D2-brane lies vertically within a D6-brane. The domain walls correspond to places where the D2-brane interpolates horizontally between the D6-branes.

control over the physics near the vacua $\phi = m_i$. Nonetheless, it is interesting to note that at least the symmetries coincide with the sine-Liouville potential. To see this, recall that the existence of two periodic collective coordinates $\sigma$ and $\chi$ in the supersymmetric case can be traced to the second term in (3.12) which first appeared as a $(\partial\sigma + \boldsymbol{\omega}\cdot\partial\boldsymbol{\phi})^2$ coupling in (3.1). In the original variables of the U(1) gauge field, this term takes the form

$$\epsilon_{\mu\nu\rho}F_{\mu\nu}\,\boldsymbol{\omega}\cdot\partial_\rho\boldsymbol{\phi}$$

which arises at one-loop from a triangle graph with fermions running in the loop (as is clear from the tell-tale presence of the $\epsilon$-symbol). We see therefore that the three-dimensional fermions provide the delicate mechanism by which the relative internal degree of freedom $\chi$ preserves its shift symmetry. If we remove the fermions in three dimensions, there is nothing to prevent the appearance of $\chi$ in the bosonic potential.

## 4. Discussion

In this, final section, I would like to discuss a few variations on the theme. We will start with a description of how these domain walls can appear in brane dynamics. This adds one further entry to the list of occurrences of the two-dimensional Euclidean black hole in critical superstring theory. We will then discuss further generalisations of the idea within field theory and add some speculations.

**A brane construction.** The brane construction for domain walls has been discussed several times in the literature [29, 15, 16] and the only new point here is the relationship to the two-dimensional black hole described in section 2. The set-up starts with the familiar D2-D6 system in IIA string theory (the M-theory lift was described in [29]). The low-energy interacting modes on a single D2-brane in the background of 3 parallel, separated D6-branes are described by the lagrangian (2.1), where the FI parameter $\zeta$ is induced by a background NS-NS B-field. This B-field induces an attractive force between the D2-brane and D6-branes, resulting in three supersymmetric vacuum states in which the D2-branes lies within one of the D6-branes. The supersymmetric domain wall solutions of the field theory appear when the D2-brane interpolates from one D6-brane to another as shown in figure 4. When the separation between the domain walls is small compared to the B-field then $k \gg 1$ and the dynamics of the domain walls is naturally given in terms of the weakly coupled black hole sigma model (1.1). When the separation grows, the weakly coupled description is in terms of the supersymmetric Liouville theory (1.2).

**Generalisations** There is a natural generalisation of the discussion in this paper to three dimensional theories with $(N+1)$ hypermultiplets. If each of these has a distinct mass $\mathbf{m}_i$, then there are $(N+1)$ vacua and one may consider the dynamics of domain walls interpolating from the first to the last. This system was studied in detail in [17, 18]. For co-linear masses $\mathbf{m}_i = (m_i, 0, 0)$, it is known that the most general solution has $2N$ collective coordinates which have the interpretation of the position and internal phase of $N$ component domain walls. The moduli space has the structure,

$$\mathcal{M}_N = \mathbf{R} \times \frac{\mathbf{R} \times \tilde{M}_{N-1}}{\mathcal{G}},$$

where the first two $\mathbf{R}$ factors parameterise the center-of-mass and overall phase of the soliton respectively. The relative kink moduli space $\tilde{\mathcal{M}}_{N-1}$ has complex dimension $(N-1)$ and is endowed with a Kähler metric. The precise form of this metric is unknown for $N > 2$, although it can be shown to have $(N-1)$ holomorphic isometries which are inherited from the $\mathrm{U}(1)^N$ flavour symmetry of the three-dimensional gauge theory. The quotient by the discrete group $\mathcal{G}$ acts only on the toric fibers and has no fixed points. Thus, $\tilde{\mathcal{M}}_{N-1}$ is a smooth, non-compact, toric Kähler manifold, and the dynamics of the classical domain walls is described by the $\mathcal{N} = (2,2)$, $d = 1+1$ non-linear sigma-model on $\tilde{\mathcal{M}}_{N-1}$.

In the other description of domain walls, one integrates out the $(N+1)$ hypermultiplets, and restricts to the Coulomb branch where the low-energy dynamics is given by (3.1) with the harmonic function $H(\phi)$ given by the natural generalisation of (3.2) to include $(N+1)$ terms. In this case, there is again a force between domain walls, resulting in a Landau-Ginzburg description of the dynamics. We leave the exact computation of this potential for future work, but the resulting theory is expected to be the Landau-Ginzburg mirror of the toric-sigma model described in [9].

Let us mention a limit in which other, more familiar, sigma models can be realised on the domain wall worldvolume. Consider, for example, the situation with 4 hypermultiplets so that we are studying a system of 3 domain walls as shown in figure 3. The tension of the $i^{\text{th}}$ domain wall is given by $T_i = \zeta(m_i - m_{i+1})$. We can consider the limit in which $m_1 \to -\infty$ and $m_4 \to +\infty$ with $m_2$ and $m_3$ kept finite. This ensures that the first and third domain wall become very heavy and hence static, fixed at some distance $L$. Meanwhile, the middle domain wall is free to move but restricted to lie between the outlying domain walls which now act as bookends. The moduli space of this system therefore takes the form of an interval parameterised by $R \in [0, L]$. Added to this, we have the internal phase $\theta \in \mathbf{S}^1$ associated with the middle domain wall. Since $\tilde{\mathcal{M}}_2$ is smooth, the submanifold describing the interactions of the middle domain wall should also be smooth. This suggests that the

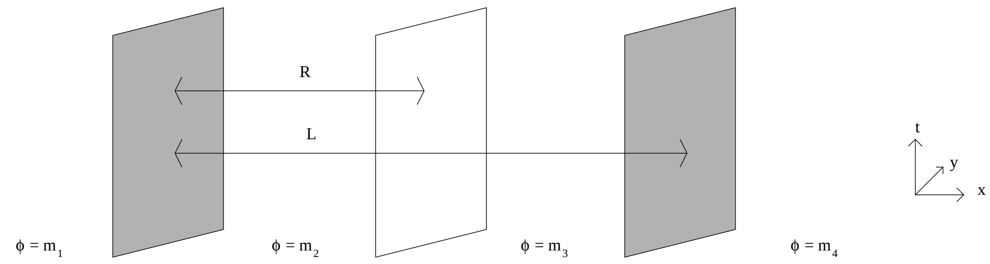


**Figure 4:** Three domain walls. When the outer two become heavy (shown by the shading) the light, middle domain wall is restricted to bounce between them. This gives a realisation of the $\mathbf{CP}^1$ sigma-model.

$\mathbf{S}^1$ is fibered over the interval in such a way that it degenerates at $R = 0$ and $R = L$, resulting in a smooth moduli space with topology $\mathbf{CP}^1$. The Kähler metric on this space is squashed compared to the round Fubini-Study metric by an amount dependent upon $L, \zeta$ and $m_i$, resulting in what is sometimes called the "sausage model" [30]. In a similar fashion, one can construct squashed $\mathbf{CP}^n$ models by considering the dynamics of $n$ light domain walls sandwiched between two heavy ones. The mirror Landau-Ginzburg theories for these sigma-models were discussed in [9, 8] and are of the $A_n$ affine Toda form. For example, in the $\mathbf{CP}^1$ case, the mirror theory has a potential of the form $V \sim \exp(-R) + \exp(R - L)$. It is clear that in our effective theory approach, such a potential will be generated by the repulsive force between the light domain wall and the two heavy bookends. Details will be provided elsewhere.

It is worth noting that, in each of the examples above, the classical theory admits domain wall solutions with arbitrary separation while, in the low-energy effective theory, the domain walls repel. One may wonder if this phenomenon is generic: do quantum effects always generate a repulsive force between domain walls in these theories? In fact they do not. To see this, we may invoke mirror symmetry of three dimensional gauge theories. Recall that the U(1) gauge theory with $N$ hypermultiplets is equivalent to the $A_{N-1}$ quiver theory, where the Higgs and Coulomb branches of the two theories are exchanged [25]. In the quiver theory, one may check that domain walls repel classically. However, after integrating out the matter multiplets, one finds that the force between the walls vanishes. Thus, in this theory, the quantum effects lead to an attractive force between the walls which precisely cancels the classical repulsive force.

Throughout this paper, we have focused on deriving the quantum duality of two-dimensional field theories from classical domain wall dynamics. However, one could turn the issue around and ask if either of the $\mathcal{N} = (2, 2)$ SCFTs described by (1.1) or (1.2) can teach us about the quantum dynamics of domain walls in the three-dimensional theory. For the bosonic CFT, both the spectrum and partition function are known [32, 33] and these calculations were recently extended to the superconformal case [34]. As well as the expected continuum of scattering states, the spectrum includes towers of discrete states with wavefunctions localised at the tip of the black hole. (Unitarity of these representations was discussed in [31]). In the bosonic theory, these states are labeled by their representation under $SL(2, \mathbb{R})$ and their momentum along the U(1) isometry, and include contributions from winding modes. From the three-dimensional perspective, the discrete states correspond to bound states of domain walls. In particular, the momentum modes of the sigma-model are related to dyonic domain walls, known as Q-kinks, which carry a global flavour charge [19, 16]. It would be of interest to make the mapping between the chiral primaries of the SCFT and BPS domain wall states more precise.

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